# The gravitational description of coarse grained microstates

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ABSTRACT: In this paper we construct a detailed map from pure and mixed half-BPS states of the D1-D5 system to half-BPS solutions of type IIB supergravity. Using this map, we can see how gravity arises through coarse graining microstates, and we can explicitly confirm the microscopic description of conical defect metrics, the $M = 0$ BTZ black hole and of small black rings. We find that the entropy associated to the natural geometric stretched horizon typically exceeds that of the mixed state from which the geometry was obtained.

## Contents

## 1. Introduction

The AdS/CFT correspondence and the counting of microstates of black holes in string theory provide a substantial amount of evidence that gravity is thermodynamic in nature, and that classical gravity arises by coarse graining over a (large number of) microstates. This point of view was elaborated in [1] for large black holes in $AdS_5$ and also for 1/2 BPS geometries that asymptote to $AdS_5 \times S^5$, following the work of [2] (for related work see also [3–12]). In the 1/2 BPS case one can develop a precise map between microstates and geometries and study the coarse graining in detail. The Hilbert space is that of $N$ free fermions in an harmonic oscillator potential, and given a state or ensemble in that Hilbert space one can associate a classical phase space density to it. This phase space density, a function on a two-dimensional plane, then completely determines the ten-dimensional metric. Using this setup one finds, as expected, that almost all states are typical, i.e. very similar to the ensemble average, and very difficult to distinguish from each other.

The purpose of this paper is to generalize and study the map between 1/2 BPS states and geometries to the $AdS_3$ case. Many aspects of this system were previously studied

in [13 – 26]. The $AdS_3$ case is interesting, as it has a somewhat richer structure, harbors many well-known solutions such as conical defect metrics, the BTZ black hole and black rings, and plays an important role in most of the microscopic derivations of black hole entropy. It is also the context in which Mathur formulated his fuzzball picture of black holes (see e.g. [20]). Though there are no macroscopic 1/2 BPS black holes in $AdS_3$ we will encounter several black-hole like features in our description. As we will show, the technical details are quite different from the $AdS_5$ case.

The relevant 1/2-BPS geometries were obtained by dualizing solutions describing classical string profiles in [15]. The classical string profile corresponds to a certain parametrized curve $\mathbf{F}(s) \subset \mathbb{R}^4$. This is not yet the most general solution, as the string can also oscillate in four other transversal directions, which we take to be a four-torus, and in addition has fermionic excitations (studied in [22]). However, in our paper, we will not consider these additional degrees of freedom.

The classical phase space of gravitational solutions is the set of curves $\mathbf{F}(s)$ of fixed "length" $N \sim \int ds |\dot{\mathbf{F}}(s)|^2$. The symplectic form on the phase space was derived in [25, 26]. From this one infers that the Fourier modes of $\mathbf{F}$ correspond to standard free bosonic string oscillators without the zero mode, with the length corresponding to the energy or $L_0$ eigenvalue of a state. The Hilbert space in question is therefore the set of states of level $N$ in the Hilbert space of four free bosons. To a state or density matrix in this Hilbert space we will associate a phase space density which is a measure on the space of loops of fixed length. This measure will then be used to construct the explicit 1/2 BPS metric. Quantizing a subset of the degrees of freedom of the metric is a familiar procedure as all minisuperspace approaches to quantum gravity use exactly the same idea. The usual complaint about minisuperspace approximations is that the approximation is not controlled, i.e. it is not the leading term of the expansion in some small parameter. The same complaint in principle also applies to our construction, though supersymmetry will make the results more robust, and we believe that in view of the similarity with e.g. the discussion of large black holes in $AdS_5$ in [1] we are still learning valuable lessons about quantum gravity despite the 1/2-BPS restriction.

This paper is organized as follows. The details of the map between microstates and geometries will be discussed in section 2. Coherent states will play an important role, and we will also discuss some subtleties associated to the choice of phase space density. Furthermore, we consider in detail the case of a circular profile, showing that for large quantum numbers the geometry given by our map differs by a small correction from that corresponding to a classical circular curve.

In section 3 we will show how we can use our setup to construct a class of generalized conical defect metrics, which include conical defect metrics with deficit angle $2\pi/n$ with $n$ integer. We will also confirm the claim of [18] that metrics with deficit angle $\alpha = 2\pi/n$ with $n$ not an integer cannot be constructed in this way, and our new metrics are the best approximation to conical deficit metrics with such values of $\alpha$.

In section 4 we study the metrics associated with various ensembles, in particular for the $M = 0$ BTZ black hole, small black rings and generic thermal ensembles in absence of a condensate. For the $M = 0$ BTZ black hole we will find that the area of the surface

(the 'stretched horizon') within which the metric differs significantly from that of the $M = 0$ BTZ black hole yields an entropy proportional to $N^{3/4}$, whereas the logarithm of the number of states scales as $N^{1/2}$. Thus, typical states are larger than what one would expect from a naive fuzzball picture of black holes, and we will discuss possible implications of this observation.

The ensembles we study are all of the form $\rho \sim \exp(-\sum_i a_i O_i)$. For these, the entropy obeys

$$dS = \sum_i a_i d\langle O_i \rangle \tag{1.1}$$

and such ensembles are therefore the most natural candidate dual descriptions of black objects that obey the first law. We will indeed find a rather universal behavior in the ensembles that we study, including the ensemble that we proposed as dual description of the small black ring in [24].

Some concluding remarks and open problems are finally given in section 5.

## 2. The map betweeen states and geometries

### 2.1 Conventions

We will follow the conventions of [26]. By dualizing a fundamental string with transversal profile $\mathbf{F}(s) \subset \mathbb{R}^4$ we obtain the following microstate geometries of the $D1 - D5$ system, written in string frame

$$ds^2 = \frac{1}{\sqrt{f_1 f_5}} \left[ -(dt + A)^2 + (dy + B)^2 \right] + \sqrt{f_1 f_5} d\mathbf{x}^2 + \sqrt{f_1/f_5} d\mathbf{z}^2$$

$$e^{2\Phi} = \frac{f_1}{f_5}, \quad C = \frac{1}{f_1} (dt + A) \wedge (dy + B) + \mathcal{C}$$

$$dB = *_4 dA, \quad d\mathcal{C} = - *_4 df_5 \tag{2.1}$$

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{ds}{|\mathbf{x} - \mathbf{F}(s)|^2} \tag{2.2}$$

$$f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|\mathbf{F}'(s)|^2 ds}{|\mathbf{x} - \mathbf{F}(s)|^2}$$

$$A = \frac{Q_5}{L} \int_0^L \frac{F_i'(s) ds}{|\mathbf{x} - \mathbf{F}(s)|^2} \tag{2.3}$$

The solutions are asymptotically $\mathbb{R}^{1,4} \times S^1 \times T^4$, $y$ parametrizes the $S^1$ which has coordinate radius $R$, and $\mathbf{z}$ are coordinates on the $T^4$ which has coordinate volume $V_4$. The Hodge duals $*_4$ are defined with respect to the four non-compact transversal coordinates $\mathbf{x}$. We can take a decoupling limit which simply amounts to erasing the 1 from the harmonic functions. The resulting metric will then be asymptotically equal to $AdS_3 \times S^3 \times T^4$.

As mentioned above, the solutions are parametrized in terms of a closed curve

$$x_i = F_i(s), \quad 0 < s < L, \ i = 1, \dots, 4. \tag{2.4}$$

and we will ignore oscillations in the $T^4$ direction as well as fermionic excitations in this paper.

The number of $D1$ and $D5$ branes is denoted by $N_1$ and $N_5$, and they are related to the charges $Q_i$ by

$$Q_5 = g_s N_5 \quad Q_1 = \frac{g_s}{V_4} N_1$$

The parameter $L$ has to satisfy

$$L = \frac{2\pi Q_5}{R}. \tag{2.5}$$

Besides, the curve has to satisfy the following relation

$$Q_1 = \frac{Q_5}{L} \int_0^L |\mathbf{F}'(s)|^2 ds \tag{2.6}$$

which reflects the fact that the original string had a fixed length. It turns out that the space of classical solutions has finite volume and therefore will yield a finite number of quantum states. Indeed, expanding $\mathbf{F}$ in oscillators:

$$\mathbf{F}(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \left( \mathbf{c}_k e^{i\frac{2\pi k}{L}s} + \mathbf{c}_k^{\dagger} e^{-i\frac{2\pi k}{L}s} \right) \tag{2.7}$$

where

$$\mu = \frac{g_s}{R\sqrt{V_4}} \tag{2.8}$$

it was first shown in [25] (see also [26]) by computing the restriction of the Poisson bracket to the space of solutions (2.1) that

$$[c_k^i, c_{k'}^{j\dagger}] = \delta^{ij} \delta_{kk'} \tag{2.9}$$

$$\left\langle \int_0^L : |\mathbf{F}'(s)|^2 : ds \right\rangle = \frac{(2\pi)^2}{L} \mu^2 N \tag{2.10}$$

$$N \equiv N_1 N_5 = \sum_{k=1}^{\infty} k \left\langle \mathbf{c}_k^{\dagger} \mathbf{c}_k \right\rangle. \tag{2.11}$$

Clearly, the number of states is finite. Using the above quantum mechanical system, we can now go ahead and construct a map between the quantum states of the theory and classical field configurations. As familiar from quantum mechanics, this map will involve the phase space distribution associated to quantum states.

## 2.2 Proposal for the map

Chiral primary operators in the dual CFT are in one to one correspondence with the states at level $N$ of a Fock space built out of 8 bosonic and 8 fermionic oscillators (or 24 bosonic oscillators if we replace $T^4$ by $K3$). Since we are only interested in fluctuations in the transverse $\mathbb{R}^4$ we will keep only four of the bosons and discard the fermions. The Hilbert space is thus spanned by

$$|\psi\rangle = \prod_{i=1}^{4} \prod_{k=1}^{\infty} (c_k^{i\dagger})^{N_{k_i}} |0\rangle, \qquad \sum k N_{k_i} = N \tag{2.12}$$

Given a state, or more generically a density matrix in the CFT

$$\rho = \sum_i |\psi_i\rangle\langle\psi_i| \qquad (2.13)$$

we wish to associate to it a density on phase space. The phase space is given by classical curves which we will parametrize as (note that $d$ and $\bar{d}$ are now complex numbers, not operators)

$$\mathbf{F}(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \left( \mathbf{d}_k e^{i\frac{2\pi k}{L}s} + \bar{\mathbf{d}}_k e^{-i\frac{2\pi k}{L}s} \right) \qquad (2.14)$$

and which obey the classical constraint (2.6).

We now propose to associate to a density matrix of the form (2.13) a phase space density of the form

$$f(d,\bar{d}) = \sum_i \frac{\langle 0|e^{\mathbf{d}_k \mathbf{c}_k}|\psi_i\rangle\langle\psi_i|e^{\bar{\mathbf{d}}_k \mathbf{c}_k^\dagger}|0\rangle}{\langle 0|e^{\mathbf{d}_k \mathbf{c}_k}e^{\bar{\mathbf{d}}_k \mathbf{c}_k^\dagger}|0\rangle}. \qquad (2.15)$$

Notice that this phase space density, as written, is a function on a somewhat larger phase space as $d,\bar{d}$ do not have to obey (2.6). We will discuss this issue in the next section and ignore it for now.

The density (2.15) has the property that for any function $g(d,\bar{d})$

$$\int\int_{d,\bar{d}} f(d,\bar{d})g(d,\bar{d}) = \sum_i \langle\psi_i| : g(c,c^\dagger) :_A |\psi_i\rangle \qquad (2.16)$$

where $: g(c,c^\dagger) :_A$ is the anti-normal ordered operator associated to $g(c,c^\dagger)$, and $\int_{d,\bar{d}}$ is an integral over all variables $d_i$. It is possible to construct other phase densities such as the Wigner measure where anti-normal ordering is replaced by Weyl ordering, or one where anti-normal ordering is replaced by normal ordering. Though apparently different, they will yield identical results if we are interested in computing expectation values of normal ordered operators. Since the theory behaves like a $1+1$ dimensional field theory this is certainly the natural thing to do in order to avoid infinite normal ordering contributions. Besides, everything we do is limited by the fact that our analysis is in classical gravity and therefore can at best be valid up to quantum corrections.

To further motivate (2.15) we notice that it associates to a coherent state a density which is a gaussian centered around a classical curve, in perfect agreement with the usual philosophy that coherent states are the most classical states. It is then also clear that given a classical curve (2.14) we wish to associate to it the density matrix

$$\rho = P_N e^{\mathbf{d}_k \mathbf{c}_k}|0\rangle\langle 0|e^{\bar{\mathbf{d}}_k \mathbf{c}_k^\dagger}P_N \qquad (2.17)$$

where $P_N$ is the projector onto the actual Hilbert space of states of energy $N$ as defined in (2.12). Because of this projector, the phase space density associated to a classical curve is not exactly a gaussian centered around the classical curve but there are some corrections due to the finite $N$ projections. Obviously, these corrections will vanish in the $N \to \infty$ limit.

Since the harmonic functions appearing in (2.1) can be arbitrarily superposed, we finally propose to associate to (2.13) the geometry

$$f_5 = 1 + \frac{Q_5}{L} \mathcal{N} \int_0^L \int_{d,\bar{d}} \frac{f(d,\bar{d})ds}{|\mathbf{x} - \mathbf{F}(s)|^2}$$

$$f_1 = 1 + \frac{Q_5}{L} \mathcal{N} \int_0^L \int_{d,\bar{d}} \frac{f(d,\bar{d})|\mathbf{F}'(s)|^2 ds}{|\mathbf{x} - \mathbf{F}(s)|^2}$$

$$A^i = \frac{Q_5}{L} \mathcal{N} \int_0^L \int_{d,\bar{d}} \frac{f(d,\bar{d})\mathbf{F}'_i(s)ds}{|\mathbf{x} - \mathbf{F}(s)|^2} \tag{2.18}$$

with the normalization constant

$$\mathcal{N}^{-1} = \int_{d,\bar{d}} f(d,\bar{d}) \tag{2.19}$$

It is interesting to contrast this approach to the results of [23]. In that paper, properties of the geometry were derived from microstates by evaluating two-point functions in the CFT. Assuming that the two-point functions do not renormalize from weak to strong coupling, this provides a direct probe of the geometry, but it is not easy to reconstruct the geometry directly. Despite this, one can see very nicely that coarse graining leads to classical gravitational descriptions, in accordance with our findings.

In [18] it was shown that the geometries corresponding to a classical curve are regular provided $|\mathbf{F}'(s)|$ is different from 0 and the curve is not self intersecting. In our setup we sum over continuous families of curves which generically smoothes the singularities. The price that one pays for this is that the solutions will no longer solve the vacuum type IIB equations of motion, instead a small source will appear on the right hand side of the equations. Since these sources are subleading in the $1/N$ expansion and vanish in the classical limit, they are in a regime where classical gravity is not valid and they may well be cancelled by higher order contributions to the equations of motion.

The distribution corresponding to a generic state $|\psi\rangle = \prod_{k=1}^\infty (c_k^{i\dagger})^{N_{k_i}} |0\rangle$ can be easily computed

$$f(d,\bar{d}) = \prod_{k,i} (d_k^i \bar{d}_k^i)^{N_{k_i}} e^{-d_k^i \bar{d}_k^i} \tag{2.20}$$

As a check, we will verify that (2.10) is satisfied. To do so, we need to come up with an operator which reproduces the left-hand side of (2.10) upon anti-normal ordering. In view of (2.9), this is relatively easy to implement, for example in $|\mathbf{F}'(s)|^2$ we simply need to replace

$$d_k^i \bar{d}_k^i \rightarrow d_k^i \bar{d}_k^i - 1. \tag{2.21}$$

We will continue to write expressions like $|\mathbf{F}'(s)|^2$ in order to not clutter the notation, but always keep in mind that shifts like (2.21) may be necessary in order to keep track of the proper normal ordering of the operator in question. Using (2.21) it is then easy to show that (2.10) is satisfied. Indeed, (2.21) is equivalent to the following condition

$$Q_1 = \frac{Q_5}{L} \mathcal{N} \int_0^L \int_{d,\bar{d}} f(d,\bar{d})|\mathbf{F}'(s)|^2 ds \tag{2.22}$$

and this is satisfied as a consequence of $\sum k N_{k_i} = N_1 N_5$. More explicitly

$$\frac{Q_5}{L}\mathcal{N}\int_0^L\int_{d,\bar{d}}f(d,\bar{d})|\mathbf{F}'(s)|^2 ds = \frac{Q_5}{L}\mathcal{N}\int_{d,\bar{d}}f(d,\bar{d})\left(\mu^2\frac{4\pi^2}{L^2}L\sum_{k=1}^{\infty}k(d_k^i\bar{d}_k^i-1)\right) =$$
$$= \mu^2\frac{4\pi^2}{L^2}Q_5\left(\sum_k kN_{k_i}\right) = Q_1. \tag{2.23}$$

To go from the first line to the second we have used the following identity

$$\int_{d,\bar{d}}(d\bar{d})^k e^{-d\bar{d}} = 4\pi\int_0^{\infty}dr\,r^{2k+1}e^{-r^2} = 2\pi k!. \tag{2.24}$$

## 2.3 Reparametrization invariance and microcanonical vs canonical

There is an important subtlety that we need to address. We wish to study the phase space of curves of fixed length. The phase space of curves of arbitrary length is very easy, it simply consists of an infinite set of harmonic oscillators. The length of the curve is measured by some operator $\hat{N}$. The constraint $\hat{N} = N$ is however first class in the language of Dirac, because $[\hat{N}, \hat{N}] = 0$ (or in classical language, the length Poisson commutes with itself). First class constraints generate a gauge invariance. In the present case, the operator $\hat{N}$ also generates a gauge invariance, which is simply the shift of the parametrization of the curve,

$$\mathbf{F}(s) \to \mathbf{F}(s + \delta s). \tag{2.25}$$

This follows immediately from the commutation relations of $\hat{N}$ with the oscillators.

Therefore, we have two possibilities: we can either not impose the length constraint, and include an extra factor $\exp(-\beta\hat{N})$ in the calculations, where we choose $\beta$ such that the expectation value of $\hat{N}$ is precisely $N$. This would be like doing a canonical ensemble, and for many purposes this is probably a very good approximation.

If we insist on fixing the length however, we also have to take the gauge invariance into account. Therefore, once we include the length constraint, it is impossible to distinguish curves whose parametrization is shifted by a constant. In particular, the expectation value of $\mathbf{F}(s)$ will always be zero, because the only meaningful quantities to compute are those of gauge invariant operators, and $\mathbf{F}(s)$ is not gauge invariant. Notice that $f_1$, $f_5$ and $A$ are gauge invariant so for those it is not a problem.

We also need to improve the map we discussed above a little bit: we need to project the measure (2.15) on loop space onto the submanifold of phase space of curves of fixed length. It is not completely trivial to determine the right measure. To get an idea we will do the simple example of two oscillators.

We consider $\mathbb{C}^2$ with the usual measure. We wish to restrict to the submanifold $N = a_1|z_1|^2 + a_2|z_2|^2$, and we wish to gauge fix the U(1) symmetry that maps $z_k \to e^{i\epsilon a_k}z_k$. What is the measure that we should use? In general, if we have a three-manifold with a U(1) action, and we gauge fix this U(1) the measure on the gauge-fixed two-manifold is simply the induced measure as long as the U(1) orbits are normal to the gauge fixed two-manifold. So if we integrate a gauge-invariant operator over the gauge fixed two-manifold,

this is the same as integrating it over the entire three-manifold, but dividing by the length of the U(1) orbit through each point. Call the length of this orbit at the point $P$ $\ell(P)$. On the three-manifold (given by $N = a_1|z_1|^2 + a_2|z_2|^2$) we have the induced measure. If we call this equation $f = 0$, then the induced measure on the three-manifold is $d^4x\delta(f)|df|$, with $|df|$ the norm of the differential $df$. So all in all we can write the integral of a gauge invariant quantity $A$ on the two-dimensional submanifold as

$$\int d^4 x A(x) \frac{\delta(f)|df|}{\ell(P)}. \tag{2.26}$$

The length of the U(1) orbit is rather tricky, for general $a_1, a_2$ the orbits do not even close. So we will assume that these numbers are integers. Then up to an overall constant that depends only on $a_i$ the length of the orbit is almost everywhere

$$\ell(P) = \sqrt{\sum a_i^2 |z_i|^2} \tag{2.27}$$

with some pathologies if some of the $z_i$ vanish.

Interestingly enough, we now see that $|df|$ and $\ell(P)$ cancel each other. Thus the only modification in the measure will be to include an extra delta function of the form

$$\delta(N - \sum_k k d_k \bar{d}_k) \tag{2.28}$$

in phase space density. As long as we integrate gauge invariant quantities this will yield the right answer. Thus, in (2.15) and in (2.20) we should include the appropriate delta function.

Inserting the delta function is just like passing from a canonical to a microcanonical ensemble. For many purposes the difference between the two is very small, and not relevant as long as we consider the classical gravitational equations of motion only. We will therefore in the remainder predominantly work in the canonical picture, commenting on the difference with the (more precise) microcanonical picture when necessary.

## 2.4 An example: the circular profile

In the following we will consider the instructive example of a circular profile. First we will compute the geometry due to a classical circular curve and then compare the result with the geometry obtained following the prescription in section 2.2. This will effectively correspond to a slightly smeared circular profile.

### 2.4.1 Classical profile

We consider the following profile

$$F^1(s) = a \cos \frac{2\pi k}{L} s, \quad F^2(s) = a \sin \frac{2\pi k}{L} s, \quad F^3(s) = F^4(s) = 0 \tag{2.29}$$

which describes a circular curve winding $k$ times around the origin in the 12-plane. In order to simplify our discussion, we focus on the simplest harmonic function $f_5$. Plugging (2.29) into (2.1) it is straightforward to compute

$$f_5 = 1 + \frac{Q_5}{\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2 + a^2)^2 - 4a^2(x_1^2 + x_2^2)}} \tag{2.30}$$

where the value of $a$ is fixed by the condition (2.6)

$$Q_1 = Q_5 \left( \frac{2\pi k}{L} a \right)^2 .$$

In order to evaluate the various integrals it will be convenient to Fourier transform the $x$-dependence. Using

$$\frac{1}{|\mathbf{x}|^2} = \frac{1}{4\pi^2} \int d^4 u \frac{e^{i\mathbf{u}.\mathbf{x}}}{|\mathbf{u}|^2} \tag{2.31}$$

we can write $f_5$ in the following equivalent way

$$f_5^{\text{clas}} = 1 + \frac{Q_5}{4\pi^2} \int d^4 u \frac{e^{i\mathbf{u}.\mathbf{x}}}{|\mathbf{u}|^2} J_0(a\sqrt{u_1^2 + u_2^2}) = \tag{2.32}$$

$$= 1 + J_0 \left( a\sqrt{-\partial_1^2 - \partial_2^2} \right) \frac{Q_5}{|\mathbf{x}|^2} \tag{2.33}$$

Writing $f_5$ in this somewhat formal way has the advantage that it can be more easily compared to the quantum expression obtained in the next section. As we will explain in section 4.1, the other harmonic functions can be obtained from the "generating harmonic function"

$$f_v = 1 + Q_5 J_0 \left( a\sqrt{\left( \frac{2\pi k}{L} v_2 + i\partial_1 \right)^2 + \left( \frac{2\pi k}{L} v_1 - i\partial_2 \right)^2} \right) \frac{1}{|\mathbf{x}|^2} . \tag{2.34}$$

For example, putting $v_1 = v_2 = 0$ immediately reproduces (2.32). The geometry can be written in a more familiar form by performing the following change of coordinates

$$\begin{aligned} x_1 &= (r^2 + a^2)^{1/2} \sin\theta \cos\varphi , & x_2 &= (r^2 + a^2)^{1/2} \sin\theta \sin\varphi \\ x_3 &= r\cos\theta \cos\psi , & x_4 &= r\cos\theta \sin\psi . \end{aligned} \tag{2.35}$$

In terms of these coordinates, the harmonic functions $f_{1,5}$ become in the near horizon limit (i.e. dropping the one)

$$f_5 = f_v|_{v=0} = \frac{Q_5}{r^2 + a^2 \cos^2\theta} , \qquad f_1 = \partial_{v^i}\partial_{v^i} f_v|_{v=0} = \frac{Q_1}{r^2 + a^2 \cos^2\theta} \tag{2.36}$$

As a consistency check, we notice that $\Box f_5$ is a delta function with a source at the location of the classical curve, to be precise $\Box |\mathbf{x} - \mathbf{F}(s)|^{-2} = -4\pi^2 \delta^{(4)}(\mathbf{x} - \mathbf{F}(s))$. One indeed finds

$$\Box f_5 = -\frac{Q_5}{4\pi^2 L} \int_0^L ds \int d^4 u \, e^{i\mathbf{u}.(\mathbf{x}-\mathbf{F}(s))} = \tag{2.37}$$

$$= -\frac{Q_5 4\pi^2}{L} \int_0^L ds \delta(x_1 - a\cos\frac{2\pi k}{L}s)\delta(x_2 - a\sin\frac{2\pi k}{L}s)\delta(x_3)\delta(x_4) = \tag{2.38}$$

$$= -4\pi^2 Q_5 \delta(x_1^2 + x_2^2 - a^2)\delta(x_3)\delta_(x_4). \tag{2.39}$$

### 2.4.2 Quantum profile

In a quantum theory it is impossible to localize wave packets arbitrarily precisely in phase space. Therefore in the quantum theory we expect to obtain a profile that is something like a minimal uncertainty Gaussian distribution spread around the classical curve. If we take the classical circular curve (2.29) then we associate to it the density matrix (2.17) and subsequently the phase space density (2.15). Working this out we find out that

$$f(d, \bar{d}) = ((d_k^1 + id_k^2)(\bar{d}_k^1 - i\bar{d}_k^2))^{N/k} e^{-\sum_{l,i} d_l^i \bar{d}_l^i}. \tag{2.40}$$

We have ignored the delta function (2.28) here and expect (2.40) to be valid for large values of $N/k$. It is therefore better thought of as a semiclassical profile rather than the full quantum profile.

According to (2.18) the harmonic function $f_5$ is now given by

$$f_5 = 1 + \frac{Q_5}{4\pi^2} \mathcal{N} \int_0^L ds \int_{d,\bar{d}} f(d, \bar{d}) \int d^4 u \frac{1}{|\mathbf{u}|^2} e^{i\mathbf{u}\cdot(\mathbf{x}-\mathbf{F}(s)) + \sum_l \frac{u^2 \mu^2}{2l}} \tag{2.41}$$

where we have used (2.31) and the constant $\sum_l \frac{u^2 \mu^2}{2l}$ appears due to the fact that we want to compute a normal ordered quantity instead of an anti-normal ordered one. The function $\mathbf{F}(s)$ depends on an infinite set of complex oscillators $d_l^i$. It can be easily seen that the contribution for the oscillators different from $d_k^1$ and $d_k^2$ cancels exactly against the normal ordering constant $u^2 \mu^2 / 2l$ mentioned above. Furthermore, by performing the following change of variables

$$d_k^\pm = \frac{1}{\sqrt{2}}(d_k^1 \pm id_k^2) \tag{2.42}$$

we see that the integral over $d_k^-$ can be easily performed and we are left with the following expression (once we express $d_k^+, \bar{d}_k^+$ in polar coordinates and integrate over the angular variable)

$$f_5 = 1 + \frac{Q_5}{4\pi^2} \int d^4 u \frac{e^{i\mathbf{u}\cdot\mathbf{x}}}{|\mathbf{u}|^2} e^{\frac{\mu^2}{4k}(u_1^2 + u_2^2)} \int_0^\infty d\rho \frac{\rho^{2N/k+1}}{(N/k)!} e^{-\rho^2} J_0(\mu \frac{1}{\sqrt{k}}\sqrt{u_1^2 + u_2^2}\rho). \tag{2.43}$$

The integral over $\rho$ can be done explicitly (see equations (A.2) and (A.3)) and we are left with

$$f_5^{\text{quantum}} = 1 + L_{N/k} \left( \frac{a^2}{4N/k} \left( \partial_1^2 + \partial_2^2 \right) \right) \frac{Q_5}{|\mathbf{x}|^2} \tag{2.44}$$

with $L_n$ the Laguerre polynomial of order $n$. Notice that, besides the approximation of ignoring the $\delta$ function (2.28) in the distribution, this result is exact in $N/k$. In order to relate both results recall that

$$L_n(x) = \sum_{m=0}^n \frac{(-1)^m n!}{(n-m)!(m!)^2} x^m$$

which allows to find the following expansion for large values of $N/k$

$$L_{N/k}(\frac{a^2 \rho^2}{4N/k}) = J_0(a\rho) - \frac{1}{N/k} \frac{a^2 \rho^2}{4} J_2(a\rho) + \dots \tag{2.45}$$

From this we see explicitly that in the limit $N/k \gg 1$ the quantum geometry coincides with the classical one. More precisely, around asymptotic infinity the harmonic functions can be written as a series expansion in $a^2/r^2$. If we focus on a given term $a^{2p}/r^{2p}$ for some fixed (but arbitrarily large) $p$ then the coefficient of such term tends to the classical coefficient as $N/k$ tends to infinity. Note, however, that for finite $N/k$ the quantum harmonic function is a finite order polynomial in $a^2/r^2$ (of degree $N/k$ ) which contains a large number of terms that are singular at the origin (and that will re-sum only in the strict $N/k$ infinite limit). These divergences at $r = 0$ may sound like a disaster, but they are actually unphysical and due to the fact that we ignored the delta function (2.28) in the distribution (2.40). Including the delta function will impose a cutoff on the $\rho$ integral in (2.43), and since all singular terms are due to the large $\rho$ behavior of the integrand in (2.43) the cutoff will remove the singularities in $f_5$.

From this discussion it is clear that we can trust our semi-classical computation provided $N/k$ is large and we do not look at the deep interior of the solution.

As for the case of the classical curve, it is instructive to compute $\Box f_5$ for this case

$$\Box f_5 = -4\pi^2 Q_5 \delta(x_3)\delta(x_4) A(x_1, x_2) \tag{2.46}$$

$$A(x_1, x_2) = \int_0^\infty d\rho \rho J_0(\sqrt{x_1^2 + x_2^2}\rho) L_{N/k}\left(\frac{a^2\rho^2}{4N/k}\right) \tag{2.47}$$

Until here we have not used any approximation. Using identity (A.3) in the appendix and approximating $\exp(\frac{a^2\rho^2}{4N/k}) \approx 1$ one obtains

$$A(x_1, x_2) = \frac{e^{-N/k \; r^2/a^2} \left(N/k \; r^2/a^2\right)^{N/k}}{(N/k - 1)! a^2} \tag{2.48}$$

with $r^2 = x_1^2 + x_2^2$. In the limit $N/k \to \infty$ $A(x_1, x_2)$ approaches $\frac{\delta(r^2/a^2 - 1)}{a^2}$ and the classical and quantum results agree. For large $N/k$ $A(x_1, x_2)$ is approximately a gaussian around $r^2 \approx a^2$ and width $1/\sqrt{N/k}$, indeed, using Stirling's formula

$$A(x_1, x_2) \approx \frac{\sqrt{N/k}}{\sqrt{2\pi}} e^{-N/k(r^2/a^2 - 1)} (r^2/a^2)^{N/k} \tag{2.49}$$

So the quantum geometry corresponds to a solution of the equations of motion in presence of smeared sources. The width of the smeared source goes to zero in the limit $N/k \to \infty$, as expected.

## 2.5 Dipole operator

To each supergravity solution we can associate a "dipole operator" defined by

$$D_{\text{sugra}} = \int_0^L |\mathbf{F}(s)|^2 ds. \tag{2.50}$$

This operator simply measures the average spread in the $\mathbb{R}^4$ plane of the curve. It will become momentarily clear why we call this a dipole operator. It is instructive to compute (2.50) for a curve dual to a generic CFT state $|\psi\rangle = \prod_{k=1}^\infty (c_k^{i\dagger})^{N_{k_i}}|0\rangle$, i.e.:

$$D_{\text{sugra}} = \mathcal{N} \int_{d,\bar{d}} \int_0^L |\mathbf{F}(s)|^2 f(d, \bar{d}) ds \tag{2.51}$$

Using the expression for the phase space density (2.20) and performing a similar computation to the one leading to (2.23) we get

$$D_{\text{sugra}} = \mu^2 L \left( \sum_k \frac{1}{k} N_{k_i} \right) \tag{2.52}$$

This happens to agree, up to normalization, with the CFT dipole operator defined in [24]. There it was shown that a thermodynamic ensemble that includes this dipole operator reproduces the thermodynamic behavior of the small black ring.

## 3. A metric for a more general conical defect?

The aim of this section is to shed some light on the claim of [18] appendix C, where it is shown that there is no conical defect metric with arbitrary opening angles. The main ingredient in the proof was the requirement of smoothness of the metric. We will try here to relax this requirement by looking at metrics obtained by coarse graining an ensemble of (possibly non- smooth) metrics.

The starting point is the supersymmetric conical metric [13, 14]

$$\frac{ds^2}{N} = -(r^2+\gamma^2)\frac{dt^2}{R^2}+r^2\frac{dy^2}{R^2}+\frac{dr^2}{r^2+\gamma^2}+d\theta^2+\cos^2\theta(d\psi+\gamma\frac{dy}{R})^2+\sin^2\theta(d\varphi+\gamma\frac{dt}{R})^2 \tag{3.1}$$

where N is the AdS radius and $2\pi\gamma$ is the opening angle. It is well known that every supersymmetric conical metric is defined by its angular momentum and $N$. The metric (3.1) is precisely identical to the metric that we would have found in the near-horizon limit in section 2.4.1 if we would also have computed the one-forms $A, B$ and evaluated (2.1), see e.g. [18] for a detailed discussion. The relation between $\gamma$ and $k$ works out to be $\gamma = 1/k$. The construction in section 2.4.1 therefore provides a construction of conical defect metrics with $k$ integer, but for $k$ non-integer the construction in section 2.4.1 fails. The reason is that the classical curve $\mathbf{F}(s)$ needs to satisfy $\int_0^L \mathbf{F}(s)ds = 0$, as $\mathbf{F}(s)$ does not have a zero-mode, and this is only true if $k$ is an integer and the curve closes.

In order to try to construct a more general conical defect metric, we first notice that according to (2.39), the source for the metric has to be contained in a circle of radius $a$ in the $x_1, x_2$-plane. The most general souce term satisfying these requirements is

$$F_1(s) = a\cos[f(s)], \qquad F_2(s) = a\sin[f(s)], \qquad F_3(s) = F_4(s) = 0 \tag{3.2}$$

where $f(s)$ is some arbitrary function which has to satisfy

$$\int_0^L e^{if(s)}ds = 0 \tag{3.3}$$

because $\mathbf{F}(s)$ does not contain a zeromode. In addition, the source (2.29) is invariant under rotations in the $x_1, x_2$-plane. To accomplish this we need to coarse grain over all

U(1) rotations of (3.2). This is most easily done by introducing polar coordinates $x_1 + ix_2 = ue^{i\varphi}, x_3 + x_4 = ve^{i\psi}$ so that the U(1) average can be expressed as

$$f_5 = 1 + \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L \frac{ds}{|ue^{i\varphi} - ae^{if(s)+i\xi}|^2 + v^2}$$

$$f_1 = 1 + a^2 \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L \frac{f'(s)^2 ds}{|ue^{i\varphi} - ae^{if(s)+i\xi}|^2 + v^2}$$

$$A = -a \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L \frac{if'(s)e^{if(s)+i\xi} ds}{|ue^{i\varphi} - ae^{if(s)+i\xi}|^2 + v^2}. \tag{3.4}$$

The constraint (2.6) on the curve now reads

$$Q_1 = a^2 \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L f'(s)^2 ds = \frac{a^2 Q_5}{L} < f'^2 > \tag{3.5}$$

Here and in the following by $< g(s) >$ we simply mean

$$< g(s) >= \int_0^L g(s)\, ds. \tag{3.6}$$

It is straight forward to evaluate the integrals in (3.4) to get

$$f_5 = 1 + \frac{Q_5}{h} \tag{3.7}$$

$$f_1 = 1 + \frac{Q_1}{h} \tag{3.8}$$

$$A = aQ_5 \frac{< f' >}{L} \frac{u^2 + v^2 + a^2 - h}{2h} d\varphi \tag{3.9}$$

with $h^2 = (u^2 + v^2 + a^2)^2 - 4a^2 u^2$. In order to put it in a form which resembles the conical defect one as much as possible, one has to make the following change of coordinates

$$u^2 = (r^2 + a^2)\sin^2\theta, \quad v = r\cos\theta \tag{3.10}$$

Using these new coordinates, the various ingredients of (2.1) become

$$f_5 = \frac{Q_5}{r^2 + a^2\cos^2\theta}$$

$$f_1 = \frac{Q_1}{r^2 + a^2\cos^2\theta}$$

$$A = \alpha \frac{a\sqrt{Q_1 Q_5}}{r^2 + a^2\cos^2\theta}\sin^2\theta d\varphi$$

$$B = -\alpha \frac{a\sqrt{Q_1 Q_5}}{r^2 + a^2\cos^2\theta}\cos^2\theta d\psi$$

$$ds_4^2 = (r^2 + a^2\cos^2\theta)(\frac{dr^2}{r^2 + a^2} + d\theta^2) + r^2\cos^2\theta d\psi^2 + (r^2 + a^2)\sin^2\theta d\varphi^2$$

$$\mathcal{C} = -\frac{Q_5 r^2 \sin^2\theta}{r^2 + a^2\cos^2\theta}d\psi \wedge d\varphi \tag{3.11}$$

where $\alpha^2 = a^2 \frac{Q_5}{Q_1}(\frac{<f'>}{L})^2 = \frac{1}{L}\frac{<f'>^2}{<f'^2>}$ is a constant introduced for later convenience. Next we rescale $r$ by a factor of $\frac{\sqrt{Q_1 Q_5}}{R}$ and define $\gamma = \alpha \frac{2\pi}{<f'>}$, and after some straightforward algebraic manipulations we end up with

$$
\frac{ds^2}{\sqrt{Q_1 Q_5}} = -\left(r^2 + \gamma^2\right)(\frac{dt}{R})^2 + r^2(\frac{dy}{R})^2 + \frac{dr^2}{r^2 + \gamma^2}
$$
$$
+ \left(d\theta^2 + \sin^2\theta(d\varphi - \alpha\gamma\frac{dt}{R})^2 + \cos^2\theta(d\psi - \alpha\gamma\frac{dy}{R})^2\right)
$$
$$
+ \frac{(1-\alpha^2)\gamma^2}{r^2 + \gamma^2\cos^2\theta}\left(\sin^2\theta d\Sigma_1^2 + \cos^2\theta d\Sigma_2^2\right)
$$
$$
\frac{C}{Q_5} = (r^2 + \gamma^2\cos^2\theta)\frac{dt}{R}\wedge\frac{dy}{R} + \frac{\alpha^2\gamma^2 - r^2\sin^2\theta}{r^2 + \gamma^2\cos^2\theta}d\psi\wedge d\varphi
$$
$$
-\alpha\gamma\left(\cos^2\theta\frac{dt}{R}\wedge d\psi + \sin^2\theta\frac{dy}{R}\wedge d\varphi\right) \tag{3.12}
$$

where we defined

$$
d\Sigma_1^2 = \sin^2\theta d\varphi^2 + \left(r^2 + \gamma^2\cos^2\theta\right)(\frac{dt}{R})^2
$$
$$
d\Sigma_2^2 = -\cos^2\theta d\psi^2 + (r^2 + \gamma^2\cos^2\theta)(\frac{dy}{R})^2
$$

This metric is a conical defect metric for $\alpha = 1$, so the question is which values of $\gamma$ are compatible with $\alpha = 1$. To analyze this, we recast the constraints on $f(s)$ for $\alpha = 1$ here

$$
\int_0^L e^{if(s)}ds = 0 \tag{3.13}
$$
$$
\left(\int_0^L f'(s)ds\right)^2 = L\int_0^L (f'(s))^2 = \left(\frac{2\pi}{\gamma}\right)^2. \tag{3.14}
$$

However, according to Schwarz's inequality,

$$
\left(\int_0^L f'(s)ds\right)^2 \le L\int_0^L (f'(s))^2 \tag{3.15}
$$

for integrable functions $f'(s)$ with equality if and only if $f'(s)$ is a constant. Thus, $\alpha \le 1$ and $\alpha = 1$ only if $f'(s) = \text{const}$. Interestingly, the metric (3.12) is in general a perfectly acceptable metric, since $\alpha \le 1$ is precisely the condition for the absence of CTC's as one can derive using the results in [27]. If $\alpha = 1$ then $f'(s) = \text{const}$ together with (3.13) imply that $f(s) = 2\pi ks/L$ for some nonzero integer $k$, and $\gamma = 1/k$. We can therefore indeed only construct conical defect metrics with $\gamma = 1/k$ and $k$ integer. For $k$ noninteger, we find a bound on $\alpha$

$$
\alpha^2 \le \left[\frac{1}{\gamma}\right]^2\gamma^2 \tag{3.16}
$$

with $[x]$ the largest integer less than or equal to $x$. Indeed, we cannot come arbitrarily close to a noninteger conical defect metric in this way.

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## 4. Thermal ensembles

In the following we consider the geometry corresponding to various thermal ensembles of interest.

### 4.1 M=0 BTZ

We start by considering the ensemble corresponding to the $M = 0$ BTZ black hole. In principle one should consider a micro-canonical ensemble with states of fixed level

$$\hat{N}|\psi\rangle \equiv \sum_k k c_k^{i\dagger} c_k^i |\psi\rangle = N|\psi\rangle$$

We will, instead, consider a canonical ensemble, since in the large $N$ limit the difference between the two should vanish. The corresponding thermal ensemble is characterized by the following density matrix[1]

$$\rho = \sum_{N_k, \tilde{N}_k} \frac{|N_k\rangle \langle N_k | e^{-\beta \hat{N}} | \tilde{N}_k \rangle \langle \tilde{N}_k |}{\mathbf{Tr} e^{-\beta \hat{N}}} \tag{4.1}$$

where $|N_k\rangle$ is a generic state labelled by collective indices $N_k$

$$|N_k\rangle = \prod_k \frac{1}{\sqrt{N_k!}} (c_k^\dagger)^{N_k} |0\rangle$$

and we have chosen a normalization so that $\langle N_k | \tilde{N}_k \rangle = \delta_{N_k, \tilde{N}_k}$. The value of the potential $\beta$ has to be adjusted such that $\langle \hat{N} \rangle = N$. It is clear that

$$\rho = \prod_n \rho_k, \qquad \rho_k = (1 - e^{-k\beta}) \sum_{n=0}^{\infty} e^{-nk\beta} |k, n\rangle \langle k, n| \tag{4.2}$$

with $|k, n\rangle = \frac{1}{\sqrt{n!}} (c_k^\dagger)^n |0\rangle$. Then the full distribution will simply be the product $f(d, \bar{d}) = \prod_k f_{d_k, \bar{d}_k}^{(k)}$ with

$$f_{d_k, \bar{d}_k}^{(k)} = (1 - e^{-k\beta}) e^{-d_k \bar{d}_k} \sum_{n=0}^{\infty} \frac{e^{-nk\beta}}{n!} (d_k \bar{d}_k)^n = (1 - e^{-k\beta}) \exp\left(-(1 - e^{-k\beta}) d_k \bar{d}_k\right)$$

We start by computing $f_5$, this is given by

$$f_5 = \frac{Q_5}{4\pi^2 L} \mathcal{N} \int d^4 u \int_0^L dr \int_{d, \bar{d}} f(d, \bar{d}) \frac{e^{\sum_k \frac{u^2 \mu^2}{2k}} e^{i\mathbf{u}.(\mathbf{x} - \mathbf{F}(r))}}{|\mathbf{u}|^2} \tag{4.3}$$

The first term in the exponential is due to the fact that we want to compute a normal ordered quantity, see the discussion around (2.16). The $d, \bar{d}$-integrals are gaussian and can easily be performed,

$$f_5 = \frac{Q_5}{4\pi^2} \int d^4 u \frac{\exp\left(-\frac{|\mathbf{u}|^2 \mu^2}{2} \sum_{k=1}^{\infty} \frac{1}{k(1 - e^{-k\beta})} + \frac{|\mathbf{u}|^2 \mu^2}{2} \sum_{k=1}^{\infty} \frac{1}{k} + i\mathbf{u}.\mathbf{x}\right)}{|\mathbf{u}|^2}. \tag{4.4}$$

---

[1]We are going to ignore the $i$-index in some equations where it does not play any role. We hope that this will not create any confusion.

In the limit $\beta \ll 1$ the first sum in the exponent can be approximated by

$$\sum_{k=1}^{\infty} \frac{1}{k(1 - e^{-k\beta})} = \sum_k \frac{1}{k} + \frac{\pi^2}{6\beta} + \mathcal{O}(\log \beta) \tag{4.5}$$

and inserting this in (4.4) we see that the divergent piece drops out, as expected, and we are left with

$$f_5 = Q_5 \frac{1 - e^{-\frac{3\beta}{\pi^2 \mu^2} x^2}}{x^2}.$$

This resembles the AdS answer $f_5 = Q_5/x^2$, but with exponentially suppressed corrections that render $f_5$ finite at $x = 0$.

The computation of $f_1$ is slightly more involved but completely analogous to that of $f_5$. It is given by

$$f_1 = \partial_{v^i} \partial_{v^i} \left( \frac{Q_5}{4\pi^2 L} \mathcal{N} \int d^4 u \int_0^L dr \int_{d,\bar{d}} f(d, \bar{d}) \frac{e^{\sum_k \left( \frac{|\mathbf{u}|^2 \mu^2}{2k} - \frac{2\pi^2 k \mu^2 |\mathbf{v}|^2}{L^2} \right)} e^{i\mathbf{u} \cdot (\mathbf{x} - \mathbf{F}(r)) + i\mathbf{v} \cdot \mathbf{F}'(r)}}{|\mathbf{u}|^2} \right) \Bigg|_{v=0}. \tag{4.6}$$

Notice that one can obtain all harmonic functions by taking appropriate $v^i$ derivatives of the quantity between parentheses and then putting $v^i = 0$. The $d, \bar{d}$ integral is again

$$\sum_{k=1}^{\infty} \frac{k}{1 - e^{-\beta k}} = \sum_{k=1}^{\infty} k + \frac{\pi^2}{6\beta^2} + \mathcal{O}(1/\beta)$$

we arrive at the following result

$$f_1 = \frac{2\pi^4 \mu^2}{3L^2 \beta^2} Q_5 \frac{1 - e^{-\frac{3\beta}{\pi^2 \mu^2} x^2}}{x^2} = Q_1 \frac{1 - e^{-\frac{3\beta}{\pi^2 \mu^2} x^2}}{x^2}$$

where in the last equality we have used the correct values for $Q_1, L, \mu$ together with the value for $\beta$ to be found below in (4.7). Using similar computations one can infer that

$$A^i = 0.$$

As we already mentioned, $\beta$ has to be fixed in such a way that $\langle \hat{N} \rangle = N$. Since the occupation number of a oscillator $c_k$ is

$$\langle \hat{N}_k \rangle = Tr \rho \hat{N}_k = \frac{e^{-k\beta}}{1 - e^{-k\beta}}$$

we find that

$$N = \langle \hat{N} \rangle = -4 \, \partial_\beta \sum_{k=1}^{\infty} \log(1 - e^{-\beta k}) = \frac{2\pi^2}{3} \frac{1}{\beta^2}$$

and therefore we fix

$$\beta = \frac{\pi \sqrt{2/3}}{\sqrt{N}}. \tag{4.7}$$

Obviously, the thermodynamic limit $N \gg 1$ corresponds to $\beta \ll 1$.

A final comment is in order. The geometry obtained differs from the *classical* $M = 0$ BTZ black hole by an exponential piece. Following [16, 20] we could put a stretched horizon at the point where this exponential factor becomes of order one, so that the metric deviates significantly from the classical $M = 0$ BTZ solution. Thus, using this criterion we find for the radius of the stretched horizon[2]

$$r_0 \approx \frac{\mu}{\beta^{1/2}} \tag{4.8}$$

with corresponding entropy proportional to $N^{3/4}$. This exceeds the entropy of the mixed state from which the geometry was obtained, the latter grows as $N^{1/2}$. We refer to the conclusions for a further discussion of this mismatch.

## 4.2 Condensate plus thermal ensemble: the small black ring

In this section we consider a slightly more complicated example, namely an ensemble consisting of a condensate of $J$ oscillators of level $q$ plus a thermal ensemble of effective level $N - qJ$. As argued in [19, 24] such an ensemble should describe (in a certain region of parameter space) a small black ring of angular momentum $J$ and dipole (or Kaluza-Klein) charge $q$.

Using the techniques developed in the previous sections we can compute the generating harmonic function for this case as well and we find

$$f_v = Q_5 L_J \left( \frac{\mu^2}{4q} \left[ \left( \frac{2\pi q}{L} v_2 + i\partial_1 \right)^2 + \left( \frac{2\pi q}{L} v_1 - i\partial_2 \right)^2 \right] \right) e^{-\frac{\mu^2 \pi^2 |v|^2}{2L^2}(N-qJ)} \frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2} \tag{4.9}$$

where $D \approx \pi\sqrt{2/3}(N - qJ)^{1/2}$ so that the geometry is purely expressed in terms of the macroscopic quantities $N, J$ and $q$.

We would like to make contact between this geometry and the geometry corresponding to small black rings studied in [24]. As we will see, in the limit of large quantum numbers both geometries reproduce the same one point functions.

In order to see this, first note that the exponential factor $e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}$ will not contribute (as it vanishes faster than any power at asymptotic infinity). Secondly notice that we can use (2.45) in order to perform the formal expansion

$$L_J \left( \frac{\mu^2}{4q} \mathcal{O} \right) = J_0(\mu\sqrt{\frac{J}{q}} \mathcal{O}^{1/2}) + \ldots \tag{4.10}$$

In order to estimate the validity of this approximation we can think of $\mathcal{O}$ as being proportional to $1/|\mathbf{x}|^2$. On the other hand $\mu\sqrt{J/q}$ can be roughly interpreted as the radius of the black ring (see [27, 24], where this parameter is called $R$). Hence the approximation is valid for large values of $J$ at a fixed distance compared to the radius of the ring, very much in the same spirit as what happened in the case of a circular profile.

---

[2] The same value is obtained if we compute the average size of the curve in $\mathbb{R}^4$, $r_0^2 \approx \langle |F|^2 \rangle$.

Using (2.34) and the above approximations it is then straightforward to compute the harmonic functions

$$f_5 = \frac{Q_5}{r^2 + \mu^2 \frac{J}{q} \cos\theta}, \qquad f_1 = \frac{Q_1}{r^2 + \mu^2 \frac{J}{q} \cos\theta} \qquad (4.11)$$

where we have used a coordinate system analogous to the one used in the discussion of the circular profile (see (2.35)). Hence in this approximation the geometry reduces exactly to that of the small black ring studied in [24].

### 4.3 Generic thermal ensemble

In the following we consider a generic thermal ensemble, where each oscillator $c_{k^i}$ is occupied thermally with a temperature $\beta_{k^i}$. We further will assume that $\beta_{k\pm}$ for the directions $1, 2$ is equal to $\beta_{k\pm}$ for the directions $3, 4$. Restricting to, say, directions $1, 2$ we are led to consider the following distribution

$$f(d, \bar{d}) = \exp\left(-(1 - e^{-\beta_{k+}})d_k^+ \bar{d}_k^+ - (1 - e^{-\beta_{k-}})d_k^- \bar{d}_k^-\right). \qquad (4.12)$$

Following the same steps as for the case of the $M = 0$ BTZ black hole we obtain

$$f_5 = Q_5 \frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2} \qquad (4.13)$$

$$f_1 = Q_1 \left(\frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2} - \frac{J^2}{4N\mu^4 D^2} e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}\right) \qquad (4.14)$$

$$A = \frac{\mu^2 JR}{2} \left(2 \frac{e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{\mu^2 D} - \frac{1 - e^{-\frac{2|\mathbf{x}|^2}{\mu^2 D}}}{|\mathbf{x}|^2}\right) (\cos^2\theta d\phi + \sin^2\theta d\psi) \qquad (4.15)$$

where $(|\mathbf{x}|, \theta, \phi, \psi)$ are spherical coordinates for $\mathbb{R}^4$ in terms of which the metric reads $ds^2 = dr^2 + r^2(d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\psi^2)$.

We see that, rather surprisingly, the geometry depends only on a few quantum numbers $N, J$ and $D$ which are given in terms of the temperatures by

$$N = 2\sum_k k \left(\frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} + \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}}\right) \qquad (4.16)$$

$$J = 2\sum_k \left(\frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} - \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}}\right) \qquad (4.17)$$

$$D = 2\sum_k \frac{1}{k} \left(\frac{e^{-\beta_{k+}}}{1 - e^{-\beta_{k+}}} + \frac{e^{-\beta_{k-}}}{1 - e^{-\beta_{k-}}}\right). \qquad (4.18)$$

As a result, the information carried by the geometry is much less than that carried by the ensemble of microstates. In fact, only $N$ and $J$ are visible at infinity while $D$ sets the size of the "core" of the geometry. We interpret this as a manifestation of the no-hair theorem for black holes. The derivation in this section assumes that the temperatures are all sufficiently

large. By tuning the temperatures, it is possible to condense one (like in the small black ring case) or more oscillators. If this happens, we should perform a more elaborate analysis, and we expect that the dual geometrical description[3] corresponds to concentric small black rings. In this case the configuration will depend on more quantum numbers than just $N, J, D$, in particular we will find solutions where the small black rings carry arbitrary dipole charge. Thus, once we try to put hair on the small black hole by tuning chemical potentials appropriately, we instead find a phase transition to a configuration of concentric small black rings, each of which still is characterized by just a few quantum numbers.

## 5. Conclusions

In this paper we proposed and studied a map from 1/2-BPS pure and mixed states in the D1-D5 system to ten-dimensional geometries that become 1/2-BPS solutions to the type IIB supergravity equations of motion in the classical limit. We restricted our attention to states associated to the bosonic fluctuations of the D1-D5 system in the four transverse non-compact directions, following the work of [15]. To construct the map we took advantage of the results of [25, 26] where the symplectic form on the appropriate space of 1/2-BPS solutions were obtained. We also crucially used the idea that coherent states should be the most classical ones. An important subtlety that we ran into is that we should work with the phase space of curves in $\mathbb{R}^4$ of fixed length proportional to $N$. This can be taken into account by inserting explicit delta-functions in the phase space densities that we found. It turns out that it is technically very difficult to work with this delta-function, so instead we decided to work with a canonical ensemble where curves of length $\ell$ are weighted with weight $\exp(-\beta\ell)$ and $\beta$ is chosen in such a way that the expectation value of $\ell$ is $N$. At large $N$ both methods should agree, but there are important differences at finite $N$. When we studied the simplest example of a circular curve, we found using the canonical ensemble[4] a metric which is very singular in the interior, but we could qualitatively argue that these singularities will disappear once we properly work in the microcanonical ensemble. It would be interesting to study this in more detail. We also saw that the quantum answer indeed corresponds to a small approximately gaussian smearing of the classical curve, whose width vanishes as $N \to \infty$.

In section 3 we have elaborated somewhat on the claim of [18] that one cannot have a conical defect metric with arbitrary opening angle. We studied ensembles of smooth metrics that resemble as much as possible conical defect metrics. It turns out that one can indeed only construct conical defect metrics with deficit angle $2\pi/k$ with $k$ integer, and explicitly gave metrics that come closest to conical defect metrics with other opening

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[3]It is not difficult to see that the harmonic functions now will take the form of multiple Laguerre polynomials with differential operator arguments acting on the generating harmonic function of the $M = 0$ BTZ solution.

[4]More precisely, we used a microcanonical approach to associate a state to a classical curve and then a canonical approach to associate a phase space density to it. If we would have used a canonical approach throughout we would have found a simple exponential phase space density which does not yield any singularities.

angles. Though we did not go out of our way to prove that our construction is the most general one, it is hard to see how one could avoid our conclusion.

In section 4 we studied different thermodynamic systems and their geometric description. This was in fact the main motivation for this work. We first looked at the $M = 0$ BTZ ensemble, and found a "quantum" metric that is exponentially close to the classical one. There are no corrections to the metric that scale as an inverse power of the radius of AdS, and therefore all one-point functions (except the one giving the total mass) vanish, in agreement with general expectations. Interestingly, we found that the natural place for the stretched horizon is at $r_0 \approx \mu\beta^{-1/2}$, with corresponding entropy $\sim N^{3/4}$. This is is different from the results in $[15-17, 20]$ where a stretched horizon was found which does yield the correct entropy of order $N^{1/2}$. This stretched horizon was found using the average wave number with a suitable occupation number, but first computing the average wave number and then the average spread of the curve is not the same as computing the average spread directly. We were therefore unable to find a natural interpretation for this stretched horizon in our approach. We tried a few other possible definitions of the stretched horizon, such as the average size of the curve, or by looking at curvature invariants of the metric, but in all cases we found the same result. It is possible that there exists a better definition, for example one based on the absorption cross section, which does yield the right entropy, and this is an interesting subject for further study which may affect other small black holes in string theory as well. Incidentally, having a larger stretched horizon with more entropy does not contradict any law of physics, but a smaller one would, since that would violate the Bekenstein-Hawking bound. A larger one could simply mean that there are many other microstates with less supersymmetry whose geometry also fits inside this particular stretched horizon.

Another example we studied was the small black ring. We managed to reproduce the geometry and one-point functions discussed in [24], thus providing further evidence that the microscopic picture of [19] is the correct one. Again, there is a stretched horizon whose associated entropy $(N - qJ)^{3/4}$ deviates from the expected result $(N - qJ)^{1/2}$ just like the $M = 0$ BTZ case.

Finally we considered more general ensembles with different temperatures for each oscillator species. It turns out that the metric is characterized by only three quantum numbers, $N, J$ and $D$, where $N$ and $J$ are the energy and angular momentum as seen at infinity, and $D$ is related to the size of the core of the solution, i.e. to the stretched horizon. This is very reminiscent of the no-hair theorem. The situation changes once we allow some oscillators to have become macroscopically occupied, i.e. form a condensate just as in Bose-Einstein condensation. In this case we expect to find a geometry corresponding to concentric small black rings, as discussed at the end of section 4.3, and this would clearly be worth exploring further.

There are several avenues for further study, such as exploring in more detail the relation between CFT correlation functions and supergravity solutions. In principle our setup predicts the one-point functions of all operators in arbitrary half-BPS states and ensembles, and it would be interesting to try to reproduce this from the CFT point of view. Another direction is to extend our approach to the 1/4-BPS case which does admit black

hole solutions with a macroscopic horizon. Unfortunately, there is to date no complete classification of 1/4-BPS solutions, but perhaps one can already make progress using the subset of solutions that have been found so far. We leave all these issues to future work.

## Acknowledgments

## A. Useful identities

$$
\begin{aligned}
\delta(x_1^2 + x_2^2 - a^2) &= \frac{1}{2} \int_0^{2\pi} ds \delta(x_1 - a\cos s)\delta(x_2 - a\sin s) \\
&= \frac{1}{8\pi^2} \int_0^{2\pi} ds \int du_1 du_2 \exp(iu_1(x_1 - a\cos s) + iu_2(x_2 - a\sin s)) \quad \text{(A.1)} \\
&= \frac{1}{2} \int_0^{\infty} d\rho \rho J_0(a\rho) J_0(\sqrt{x_1^2 + x_2^2}\rho)
\end{aligned}
$$

$$
\int_0^{\infty} d\rho \frac{\rho^{2N+1}}{N!} e^{-\rho^2} J_0(A\rho) = \frac{1}{2} {}_1F_1(1+N, 1; -A^2/4) \quad \text{(A.2)}
$$

$$
L_N(x) = \frac{e^x}{N!} \int_0^{\infty} e^{-t} t^N J_0(2\sqrt{tx}) dt \quad \text{(A.3)}
$$

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